Matrices and the Linear Complementarity Problem*

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ABSTRACT

The paper is a collection of results on the linear complementarity problem (q, M). The results are stated in terms of the matrix M.

1. INTRODUCTION

The linear complementarity problem (q, M) is: given a vector $q \in R^n$ and a matrix $M \in R^{n \times n}$, find a nonnegative vector z such that $w = q + Mz$ is also nonnegative and w and z are orthogonal.

The problem arises in various applications, including convex quadratic programming (e.g. Cottle and Dantzig [4]), bimatrix games (e.g. Lemke [30]), fluid mechanics (e.g. Cryer [12]), and solution of systems of piecewise linear equations (e.g. Eaves and Scarf [14]).

In this talk we survey existence and uniqueness results stated in terms of the matrix M. We concentrate on five classes of matrices. Three of them are of general interest in matrix theory. These are the classes denoted by K, P, and Z by Fiedler and Ptak [16]. The other two, L and Q, are of interest in connection with the linear complementarity problem. We describe the main and easy to state results. Details and additional results can be found in the references and in particular in the surveys by Eaves [13] and Lemke [32, 33] and in Chapter 10 of [1]. A promising survey by Cottle and Lemke [7] is still under preparation. I wish to express my thanks to Professor Lemke for suggesting some of the references and for interesting discussions.

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The paper is concluded with some questions concerning the relation between cones, matrices, and the linear complementarity problem.

2. Q-MATRICES

A matrix M is a *Q-matrix*, $M \in Q$, if for each vector q of the same order, the linear complementarity problem (q, M) has a solution.

A sufficient condition for $M \in Q$ is (Karamardian [27]): $M \in Q$ if the system

$$x > 0, \qquad t \geq 0,$$

$$x_i > 0 \quad \Rightarrow \quad (Mx)_i + t = 0, \tag{$*$}$$

$$x_i = 0 \quad \Rightarrow \quad (Mx)_i + t \geq 0$$

is inconsistent.

This condition is a special case of an existence theorem of Karamardian for the nonlinear complementarity problem: given a function $f: R^n \to R^n$, find $z \geq 0$ such that $f(z) \geq 0$ and z and $f(z)$ are orthogonal.

The condition is not necessary, for

$$M = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \in Q,$$

but the system $(*)$ is satisfied by $x^T = (1,0,0)$ and $t = 1$. However, for nonnegative matrices it becomes (Murty [42]) a sufficient and necessary condition: $M \geq 0$ is a Q-matrix if and only if $m_{ii} > 0$ $\forall i$. [Note that if $m_{ii} = 0$, $x_i = 1$, $x_j = 0$ $(j \neq i)$, and $t = 0$ satisfy $(*)$.]

In concluding the section we want to mention a recent paper on Q-matrices by Kelly and Watson [28].

3. P-MATRICES

Much better characterized is a subset of Q of those matrices M for which (q, M) has a unique solution. These matrices play an important role in other applications of matrix theory, notably to economics. The following (see e.g. [1]) is a list of several characterizations of P-matrices. The last two are stated in the context of linear complementarity theory.

The following conditions on a matrix M are equivalent:

(1) The principal minors of M are positive.

(2) For every nonzero vector x there exists an index i such that $x_i(Mx)_i > 0$.

(3) For every nonzero vector x there exists a positive diagonal matrix D such that $x^T MDx > 0$.

(4) Same as (3), with nonnegative diagonal replacing positive diagonal.

(5) The real eigenvalues of the principal submatrices of M are positive.

(6) For every signature matrix S ($s_{ii} = \pm 1$; $s_{ij} = 0$, $i \neq j$) there exists a positive vector x such that $SMSx$ is positive.

(7) For every vector q of the same order as M, the linear complementarity problem (q, M) has a *unique* solution.

(8) The problem (q, M) has a unique solution for every q which is a column of M, $-M$, or I and for the vector e all of whose entries are ones.

The equivalence of the first four conditions is due to Fiedler and Ptak [16]. Condition (2) is also due to Gale and Nikaido [17]. Condition (5) is due to Ostrowski [44]. Condition (6) goes back to Samelson, Thrall, and Wesler [48]; see also Murty [42]. Condition (8) is due to Tamir [49].

Additional references on P-matrices in connection with variants of the linear complementarity problem include Kaneko [25,26], Kostreva and Habetler [29], and Murty [43].

4. Z-MATRICES

For a vector q and a matrix M of the same order, consider the set $X(q, M) = \{z \geqslant 0 : q + Mz \geqslant 0\}$. A point x is a *least element* of a set X if $x \in X$ and $y \in X \Rightarrow x \leqslant y$. A matrix M is a Z-matrix ($M \in Z$) if its off diagonal entries are nonpositive (Fiedler and Ptak [16]). All these concepts are related by the following result of Tamir [49]:

$M \in Z$ if and only if for any q such that $X(q, M)$ is not empty, $X(q, M)$ has a least element which solves (q, M).

Additional references on linear complementarity problems when $M \in Z$ include Chandrasekaran [2], Mohan [38,39], and Saigal [47].

The theory of least elements is intimately related to the study of q and M for which (q, M) is equivalent to a linear program. See Mangasarian [34–37], Cottle and Pang [8,9], and Pang [45,46].

5. K-MATRICES

These are the nonsingular M-matrices, the intersection of the classes P and Z. Combining the characterizations of the previous sections yields a result of Cottle and Veinott [11]: $M \in K$ if and only if for every q of the same order as M, $X(q, M)$ has a least element which solves (q, M).

An extensive list of characterization of matrices in K, given that they are in Z, is given in [1]. Here we mention two such characterizations which are stated in terms of linear complementarity problems.

The following conditions on a Z-matrix are equivalent:

(1) $M \in K$.
(2) $M \in Q$.
(3) $(0, M)$ and (e, M) have only the trivial solutions.

The second condition is due to Mohan [38], and the third to Kaneko [24].

Characterizations of matrices in K, given that they are in P, are given by Cottle [3] and Kaneko [24] in terms of linear complementarity problem with upper bounds and parametric linear complementarity problem, both arising in certain questions in structural mechanics.

For solving linear complementarity problems with $M \in K$ see [10].

6. L-MATRICES

Many algorithms for solving linear complementarity problems are variants of Lemke's complementary pivot algorithm [31].

In applying this algorithm to (q, M) one adds a positive column M^0 and a variable z_0 and solves

$$W = M^0 z_0 + Mz + q, \qquad w \geqslant 0, \quad z_0 \geqslant 0, \quad z \geqslant 0, \quad w^T z = 0,$$

by pivoting. The algorithm may terminate with $z_0 = 0$ or with $z_0 > 0$. In the first case one gets a solution of (q, M). The second case may or may not indicate that (q, M) does not have a solution. Let L denote the class of matrices M with the property that for all q the termination of Lemke's algorithm with $z_0 > 0$ means that (q, M) has no solution.

For a P-matrix the algorithm terminates in the solution, so $P \subset L$ (Cottle and Dantzig [4]). Also $Z \subset L$ (Saigal [47]). Eaves [13] showed that matrices with positive entries on the diagonal and nonnegative entries above the diagonal belong to L. See also Garcia [18].

A matrix M is copositive if $u \geqslant 0 \Rightarrow u^T M u \geqslant 0$ (see e.g. [41], [22], [5]). These matrices are important also in combinatorics [20] and in control theory [23]. (Here the matrices are symmetric.) Two subclasses of copositive matrices are important in studying linear complementarity problems:

$$C^+ = \{M: M \text{ is copositive and}$$

$$u \geqslant 0,\ u^T M u = 0 \Rightarrow (M + M^T) u = 0\}$$

and

$$\mathrm{SC} = \{M: u > 0 \Rightarrow u^T M u > 0\}.$$

Concerning these classes one has $\mathrm{SC} \subset Q \cap L$ (Cottle and Dantzig [4]) and $C^+ \subset L$ (Lemke [31]).

A related reference on L-matrices is Evers [15].

7. CONES, MATRICES, AND THE LINEAR COMPLEMENTARITY PROBLEM

An active research area in matrix theory deals with classes of matrices defined with reference to some closed convex cone K. For example,

$$\pi(K) = \{B: BK \subseteq K\} \qquad \text{(e.g. [1])},$$

$$Z(K) = \{A: A = \alpha I - B,\ B \in \pi(K)\},$$

$$K(K) = \{A: A = \alpha I - B,\ B \in \pi(K),\ \alpha > \rho(B)\} \qquad [21],$$

where $\rho(B)$ denotes the spectral radius of B.

Some insight into the structure of these classes may be obtained through a generalized linear complementarity problem, (K, q, M): Given a closed convex cone K, a vector q in R^n, and a matrix M in $R^{n \times n}$, find a vector z in K such that $w = q + Mz \in K^* = \{y: x \in K \Rightarrow x^T y \geqslant 0\}$ and $w^T z = 0$. With reference to this problem we can consider the classes

$$Q(K) = \{M: (K, q, M) \text{ has a solution for any } q\}$$

$$P(K) = \{M: (K, q, M) \text{ has a unique solution for any } q\}.$$

It may be interesting to find out how the classes $Q(K)$ and $P(K)$ relate to $Z(K)$ and $K(K)$. Morè [40] showed that if M is positive definite, then it is in $P(K)$ for any K. Habetler and Price [19] showed that if M is K-strictly copositive ($0 \neq u \in K \Rightarrow u^T M u > 0$; see e.g. [22]), then $M \in Q(K)$.

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